Lecture 24

CS 6380 Artificial Intelligence

First-Order Logic: Truth and Inference

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Some material is based on lecture slides for CSE 571: Artificial Intelligence by Siddharth Srivastava, ASU.

First-Order Logic: Syntax

- Constants $Chennai, Madras, AI, IITM, Berkeley, Alfred$, 1, 2.., 100, ...
- Predicates $Studies(x, y), HasHotSummers(x), GoodAt(x, y), Attends(x, y)$
- Functions $Location(x), +$

Vocabulary

- Variables $x, y, a, b, ...$
- Connectives $\neg, \wedge, \vee, \rightarrow$
- Equality $=$
- Quantifiers $\forall, \exists$

Atomic Sentences in FOL

- Simple sentences: atomic sentences (a relation with constants or function application terms for arguments)
  - Studies(Alfred, AI)
  - Studies(DaughterOf(Alfred), AI)
  - HasHotSummers(Chennai)

- General form of atomic sentences
  - $Predicate(term_1, term_2, \ldots)$
  - $term_1 = term_2$
- Terms:
  - $constant, or$
  - $variable, or$
  - $function(term_1, term_2, \ldots)$

Complex Sentences

- We can combine atomic sentences as in propositional logic

$$\neg S, \quad S_1 \wedge S_2, \quad S_1 \vee S_2, \quad S_1 \Rightarrow S_2, \quad S_1 \Leftrightarrow S_2$$

- $Studies(Alfred, AI) \rightarrow GoodAt(Alfred, AI)$
- $(Location(IITM) = Chennai) \vee (Location(IITM) = Madras))$

...

Semantics: Welcome to the Mathematical Multi-Verse

- As in propositional logic, truth of FOL sentences is evaluated w.r.t. a model
- A model for FOL has much more information
  - A *universe* U of elements
  - An *interpretation* I that maps
    - Each *constant symbol* in the vocabulary to an element in U
    - Each *predicate symbol* in the vocabulary to a relation over U (of the same arity)
    - Each *function symbol* in the vocabulary to a function over U
- An interpretation *interprets* functions, predicates and constants in the vocabulary with properties of the universe

$V: R = \{Studies(x, y)\}; F = \{Location(x)\}; C = \{Alfred, AI, Madras, IITM\}$

$Model$:

$Universe = \{FetchRobot, AIBook, BiologyBook, CityOfChennai, MadrasOregon, Univ1\}$

$Interpretation$:

$[Location( )]_M = ?$ $[Studies( )]_M = ?$

$[Alfred]_M = ?; [AI]_M = ?; [Madras]_M = ?; [IITM]_M = ?$

FetchRobot



$[Studies(Alfred, AI)]_M = ?$

MadrasOregon



CityOfChennai

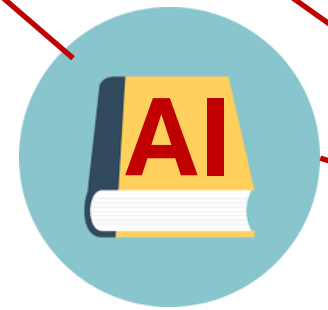
$Holding$

ℓ

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AIBook



BiologyBook



Univ1

First-Order Logic: Fundamental Concepts

- Key takeaways:
  - Vocabulary is used to write sentences
  - Models (universe + interpretation) define meaning
  - Relations, constants, functions make the language much more expressive

In addition, FOL has supports specialization and generalization through quantifiers

Quantifiers In First-Order Logic

- Key takeaways:
  - Vocabulary is used to write sentences
  - Models (universe + interpretation) define meaning
  - Relations, constants, functions make the language much more expressive

- But what about "all students at IIT Madras are good in AI"?

$$\forall x \left(Attends(x, IITM) \rightarrow GoodAt(x, AI) \right)$$

Semantics:

$\forall x \ \phi(x)$ is true in a model $M = < U, I >$ iff $\phi(c)$ is true in M for all $c \in U$

Quantifiers in First-Order Logic

- Constants: {DarkLord, HarryPotter, Dobby, Griphook}
- Predicates: {IsGoblin, IsElf, IsStick, Likes}

- Express in FOL:

The Dark Lord likes all sticks

The Dark Lord likes some sticks

The Dark Lord likes a unique stick

Interpretations

DarkLord



Dobby



IsElf() = True

Griphook



IsGoblin() = True

Statements

$[Goblins\ like\ gold] = ?$

$[DarkLord\ likes\ sticks] = ?$

Truth Under Quantification

- To assess truth of "Dark Lord likes some sticks", answer:

  Does there exist a stick in this model that the Dark Lord likes?

Universe



DarkLord





IsElf() = True

Griphook



IsGoblin() = True

Statements

$[Goblins\ like\ gold] = ?$

$[DarkLord\ likes\ sticks] = ?$

Truth Under Quantification

- To assess truth of “Dark Lord likes all sticks”, answer:

  is there a stick in this model that the Dark Lord does not like?

Universe



DarkLord





IsElf() = True

Griphook



IsGoblin() = True

Statements

$[Goblins\ like\ gold] = ?$

$[DarkLord\ likes\ sticks] = ?$

Truth in FOL

- $\forall x\ \varphi(x)$ is True in a model M iff $\varphi(u/x)$ is true for every u in the universe of M
  - Roughly the same as $\Lambda_{u \in U_M}\ \varphi(u)$
  - Not the same when the universe is infinite
- $\exists x\ \varphi(x)$ is True in a model M iff $\varphi(u/x)$ is true for some u in the universe of M
  - Roughly the same as $\mathrm{V}_{u \in U_M}\ \varphi(u)$
  - Not the same when the universe is infinite
- Only defined for sentences where all variables are bound by $\forall$ or $\exists$

- Validity is a stronger (and more useful) notion
  - Closer to what we might consider "fact"
- A sentence is valid if it is true in all possible models ($M \vDash \varphi$ holds for all M)
  - $\vDash \varphi$

Moodle Discussions!

- Constants $Chennai, Madras, AI, IITM, Berkeley, Alfred$, 1, 2.., 100, …
- Predicates $Studies(x, y), HasHotSummers(x), GoodAt(x, y),$ $Attends(x, y)$
- Functions $Location(x), +$

Vocabulary

After Class Task:

Post examples of sentences in English where an LLM (CharGPT, Claude, Gemini, etc.) fails to create a correct logical translation, or vice versa!

Truth in the Real World

- First-Order Logic has very clear definitions of truth in a given model
- What about truth in the real world? How would you assess whether a certain claim/statement is true?
- Hint: how do scientists evaluate theories?

Properties of Quantifiers

- *Note:*
  - $\forall x\ \forall y$ is the same as $\forall y\ \forall x$
  - $\exists x\ \exists y$ is the same as $\exists\ y\ \exists x$
  - $\forall x \exists y$ is NOT the same as $\exists y\ \forall x$
  - $\exists y\ \forall x\ In(x, y)$
    - All items are in one "container"
  - $\forall x\ \exists y\ In(x, y)$
    - Every item is in some "container"

$\forall x\ \varphi(x)$ is True in a model M iff $\varphi(u/x)$ is true for every u in the universe of M

$\exists x\ \varphi(x)$ is True in a model M iff $\varphi(u/x)$ is true for some u in the universe of M

Summary: Evaluating Truth

- Model theory: determine truth using models
- "φ is true under model M"

$$M \models \varphi$$

- "φ is true under any model where the set of statements T is true"

$$T \models \varphi$$

In the standard model of natural numbers, N, sum is no lesser than a part:

$$N \models \forall x, y \quad sum(x, y) \geq x$$

Inference in First-Order Logic

- Recall, the big draw of mathematical logic:
  - Can we create automated inference procedures that can derive *all true implications of a theory?*
- Recall: Propositional logic suffers from limitations similar to factored representations
  - Cannot decompose and evaluate: "Does the DarkLord like all sticks" vs "Does the DarkLord like some sticks"
  - BUT it has an automated inference procedure
- Can we extend this to first-order logic?



The Lure of Mathematical Logic, Pt 2

- Remember, we wanted to design a system to derive implications of a theory T
- E.g., under my current theory, if the car moves forward at 15mph, will there be an accident?

- ϕ is valid iff it holds in all possible models (for all M, $M \models \phi$)
- A theory T semantically entails ϕ $(T \models \phi)$ iff $T \rightarrow \phi$ holds in all models
  - For every M, $\mathrm{M} \models T \rightarrow \phi$

The Lure of Mathematical Logic, Pt 2

- Proof systems reduce the quest for true implications to an algorithm
  - Without a proof system, to determine whether a theory implies a property is true, one would have to test the property under all possible models

- This set off a quest for an accurate proof system for first-order logic (FOL)
- The idea was to express a theory of mathematics in FOL, start the computer, take a vacation

FO Axioms

This was tantalizing to logicians, mathematicians and early computer scientists:

If we can write axioms of mathematics in FOL,
we can use the proof system to algorithmically resolve all open conjectures
→ *TONS of applications*

Led to Automated Theorem Provers (the kind that proved the 4CT)

Launched a quest to axiomatize mathematics using FOL

Good News: There are many proof systems for FOL!!

$$\frac{}{\Gamma, \Phi \longrightarrow \Phi}\,(Ax)$$

$$\frac{\Gamma \longrightarrow \Phi \qquad \Gamma \longrightarrow \Phi'}{\Gamma \longrightarrow \Phi \wedge \Phi'}\,(\wedge I)$$

$$\frac{\Gamma \longrightarrow \Phi_1 \wedge \Phi_2}{\Gamma \longrightarrow \Phi_i}\,(\wedge E_i) \quad (i = 1, 2)$$

$$\frac{\Gamma, \Phi \longrightarrow \Phi'}{\Gamma \longrightarrow \Phi \Rightarrow \Phi'}\,(\Rightarrow I)$$

$$\frac{\Gamma \longrightarrow \Phi \Rightarrow \Phi' \qquad \Gamma \longrightarrow \Phi}{\Gamma \longrightarrow \Phi'}\,(\Rightarrow E)$$

$$\frac{\Gamma \longrightarrow \Phi_i}{\Gamma \longrightarrow \Phi_1 \vee \Phi_2}\,(\vee I_i) \quad (i = 1, 2)$$

$$\frac{\Gamma \longrightarrow \Phi \vee \Phi' \qquad \Gamma, \Phi \longrightarrow \Phi'' \qquad \Gamma, \Phi' \longrightarrow \Phi''}{\Gamma \longrightarrow \Phi''}\,(\vee E)$$

$$\frac{\Gamma, \Phi \longrightarrow \mathbf{F}}{\Gamma \longrightarrow \neg\Phi}\,(\neg I)$$

$$\frac{\Gamma \longrightarrow \Phi \qquad \Gamma \longrightarrow \neg\Phi}{\Gamma \longrightarrow \Phi'}\,(\neg E)$$

$$\frac{\Gamma \longrightarrow \Phi}{\Gamma \longrightarrow \neg\neg\Phi}\,(\neg\neg I)$$

$$\frac{\Gamma \longrightarrow \neg\neg\Phi}{\Gamma \longrightarrow \Phi}\,(\neg\neg E)$$

+

$$\frac{\Gamma \longrightarrow \Phi[y/x]}{\Gamma \longrightarrow \forall x \cdot \Phi}\,(\forall I) \quad (y \text{ not free in } \Gamma)$$

$$\frac{\Gamma \longrightarrow \forall x \cdot \Phi}{\Gamma \longrightarrow \Phi[t/x]}\,(\forall E)$$

$$\frac{\Gamma \longrightarrow \Phi[t/x]}{\Gamma \longrightarrow \exists x \cdot \Phi}\,(\exists I)$$

$$\frac{\Gamma \longrightarrow \exists x \cdot \Phi \qquad \Gamma, \Phi[y/x] \longrightarrow \Psi}{\Gamma \longrightarrow \Psi}\,(\exists E) \quad (y \text{ not free in } \Gamma)$$

This is one such example

More Good News

- In 1929, Kurt Gödel delivered the good news:
  - Some FOL proof systems (e.g., Gentzen's) are sound and complete
  - $T \vdash \varphi$: the proof system can derive φ from T
  - Gödel proved that there is a proof system such that $T \vDash \varphi$ iff $T \vdash \varphi$
  - Yay! Systematic method for churning out all true statements of mathematics (and consequently, all physics, engineering and later, AI...)



Good News, Bad News

- But in 1931, Kurt Gödel provided some bad news:
  - There is no recursively enumerable* set of axioms T such that for all expressions φ, $T \vdash \varphi$ iff $N \vDash \varphi$
  - Number theory cannot be axiomatized ☹
- In 1937, Alan Turing proved that there is no general method for deciding whether or not a (Turing) machine will halt on a given input



*possible to generate systematically (even infinite sets are ok)

Where Does This Leave Us?

- Turing machines have their limits
  - (and they formalize *all effective procedures*)
- Computers (restricted forms of Turing Machines) have their limits
  - But that doesn't stop them from being useful
- First-Order Logic is used in AI for representation, modeling
- In some cases, also for proving statements
  - Most theories do not include Number Theory
- SAT Solvers: automated methods for determining whether a propositional formula φ is valid (φ "valid" iff $\neg\varphi$ is not satisfiable)
  - Used in many applications including verification of systems

Applications of Logic in AI

- Automated reasoning
  - Deterministic: SAT solvers, theorem provers
  - Probabilistic: relational models e.g., Markov logic networks, Problog, BLOG, ….
    - Most systems use logic for representations, but some also utilize variants of proof systems
    - ”Lifted inference”
- Automated planning/sequential decision-making
  - Deterministic: efficient representation, automatic computation of heuristics
  - Under uncertainty (MDPs):efficient representation, symbolic methods for computing policies
  - Under partial observability (POMDPs): efficient representation, managing situations with unknown numbers of objects
- Learning
  - Decision-tree style rule learners with relational decision nodes help in generalization
- Permeates, and in many cases, initiated numerous CS subfields
  - Programming languages, model checking (software and hardware), databases, …

AI News!!

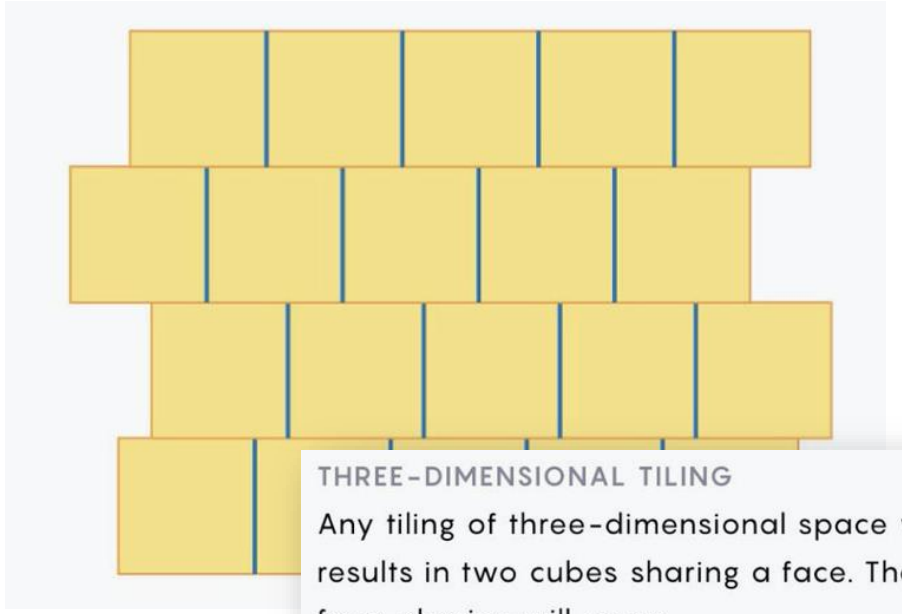
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CORONAVIRUS FAQS FACE MASKS ESSENTIALS HI

A Fleet of Computers Helps Settle a 90-Year-Old Math Problem

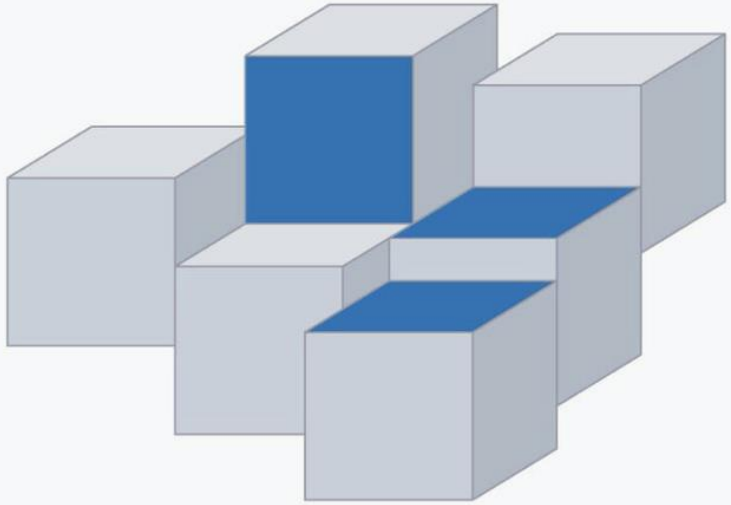
By translating Ott-Heinrich Keller's conjecture into a computer-friendly search, researchers confirmed a conjecture about seven-dimensional space.





THREE-DIMENSIONAL TILING

Any tiling of three-dimensional space with equal-sized cubes results in two cubes sharing a face. The blue squares are where face-sharing will occur.



Proved using AI (Automated Reasoning):

- Express conjecture using Mathematical Logic;
- Use SAT solver for deduction