Lecture 21

CS 6380 Artificial Intelligence

Propositional Logic: Automating Reasoning

Department of Computer Science and Engineering

Indian Institute of Technology Madras (IITM), India



Some material is based on lecture slides for CSE 571: Artificial Intelligence by Siddharth Srivastava, ASU.

Recap

Propositional Logic: Syntax

- Syntax: How to state something
- Semantics: What a statement means

The proposition symbols P_1, P_2 etc are sentences

If S is a sentence, $\neg S$ is a sentence (negation)

If S_1 and S_2 are sentences, $S_1 \wedge S_2$ is a sentence (conjunction)

If S_1 and S_2 are sentences, $S_1 \vee S_2$ is a sentence (disjunction)

If S_1 and S_2 are sentences, $S_1 \Rightarrow S_2$ is a sentence (implication)

If S_1 and S_2 are sentences, $S_1 \Leftrightarrow S_2$ is a sentence (biconditional)

Propositional Logic: Semantics

P_1 = "It rains a lot in Tempe"

P_2 = "The university in Tempe has more than 100 AI students"

Models are like possible worlds/interpretations

- Model M_1
  - $[P_1]_{M_1} = False$
  - $[P_2]_{M_1} = True$
- Another model: M_2
  - $[P_1]_{M_2} = True$
  - $[P_2]_{M_2} = False$



The vale of Tempe (note: no university)

- A model assigns a truth value to every proposition

Validity in Propositional Logic

- In mathematical logic, validity supersedes truth
- A sentence is valid if it is true in all possible models
  - You are familiar with truth tables: every row in a truth table is a model

- Scientific notions of truth try to achieve this by conducting experiments
  - Like testing under different models, but sometimes not possible to enumerate all possibilities

Theories in Mathematical Logic

- E.g., P_1: "AI is a hot topic"; P_2: "A university in town is good in AI"

"P_3: town has hot summers"

Theory = collection of logical sentences (a conjunction of formulas)

E.g.,

$$T = \{P_1 \wedge P_3 \rightarrow P_2\}$$

- A theory T is said to be consistent with a model M iff $M \models T$
- A theory is said to be logically consistent iff it doesn't imply a contradiction
  - A contradiction is any statement of the form $P \wedge \neg P$

Automating Reasoning

Can We Automate Deductive Reasoning?

- Can we build an automated inference procedures to derive *all true implications of a theory??*
- Huge applications in science, mathematics and AI
  - E.g., if we can write down a logical theory of mathematics, all theorems can be computed automatically!
  - Newton's laws of motion + optics, in a self-driving car: compute all safe paths!

Can We Automate Deductive Reasoning?

- Can we build an automated inference procedures to derive *all true im*
- Hu
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  - aths!

That's the lure of mathematical logic!

In Class Exercise. More Puzzles Like This One

- Same drill as the solar-powered car. Read each pair of sentences, work out the answer, then notice what pattern you used
- If there is a fest at IIT Madras, the hostels get a late curfew. Tonight there is a fest. Do the hostels get a late curfew
- If the shuttle is late, students miss the first class. The shuttle is on time today. Do students miss the first class
- If it is exam season, the library stays open all night. It is exam season. Is the library open all night
- Try answering all three in under a minute, without drawing a truth table
- Which of these three could you not answer with full confidence, and why?

If there is a fest at IIT Madras, the hostels get a late curfew. Tonight there is a fest. Do the hostels get a late curfew

- Let P be "there is a fest," Q be "the hostels get a late curfew."
- Given: $P \rightarrow Q$, and P.
- By Modus Ponens ($P \rightarrow Q$, P, therefore Q): **Q is true.**
- Hence, the hostels get a late curfew.

If the shuttle is late, students miss the first class. The shuttle is on time today. Do students miss the first class

- Let P be "the shuttle is late," Q be "students miss the first class."
- Given: $P \rightarrow Q$, and $\neg P$ (the shuttle is on time, so P is false).
- By Modus Ponens ($P \rightarrow Q$, P, therefore Q)
- Since $\neg P$, you cannot conclude about Q.

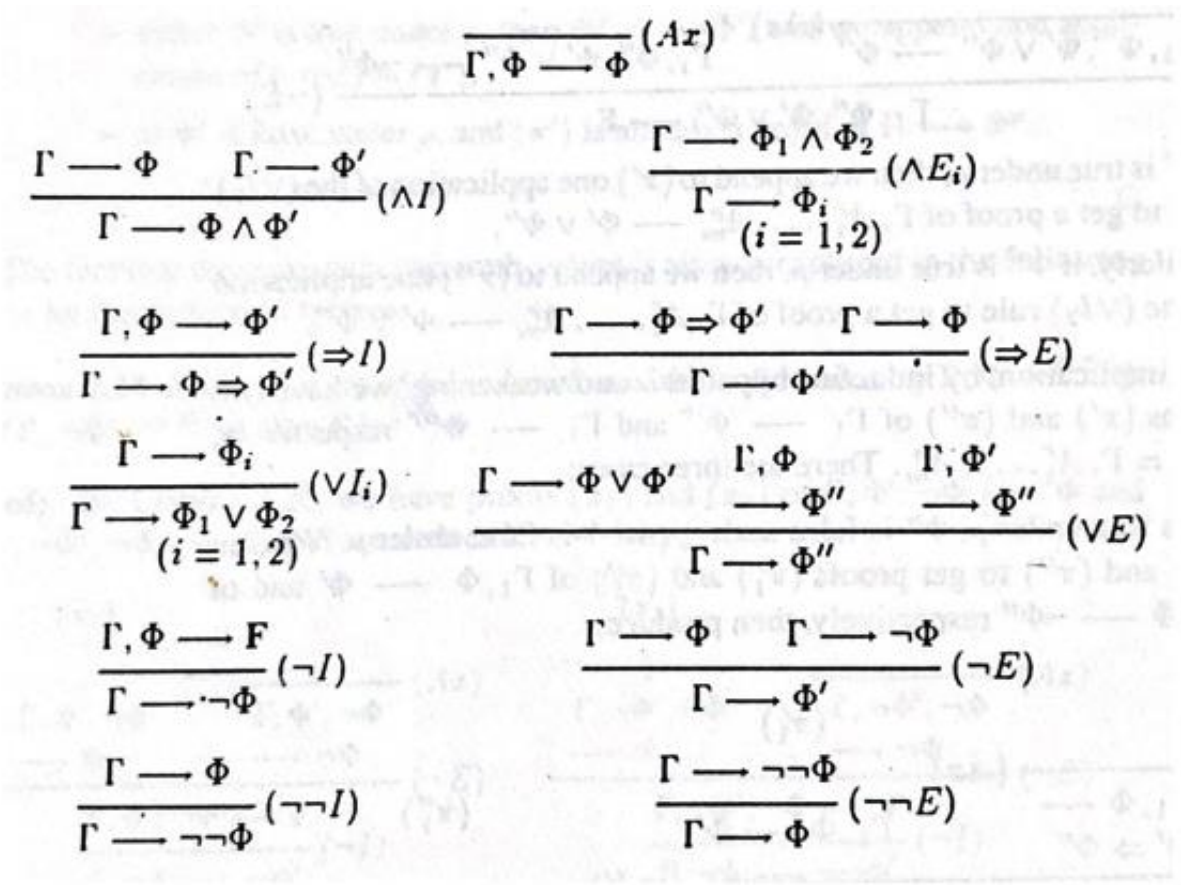
If it is exam season, the library stays open all night. It is exam season. Is the library open all night

- Let P be "it is exam season," Q be "the library stays open all night."
- Given: $P \rightarrow Q$, and P.
- By Modus Ponens ($P \rightarrow Q$, P, therefore Q): **Q is true.**
- Hence, the library is open all night.

The Pattern You Just Used

- The solar car question and the first two IIT Madras questions all had the same shape
- From “if P then Q” and “P”, conclude “Q”. This pattern is called Modus Ponens
- The shuttle question was different, since it told you the shuttle was on time, making the antecedent false. Modus Ponens needs the antecedent itself to be true, so that puzzle could not be answered the same way
- Every one of you found these answers instantly, without writing a truth table. Good news, that means the pattern is mechanical
- If a pattern is mechanical, a machine can apply it too. That is the whole idea behind a proof system
- A proof system is just a small set of mechanical patterns like Modus Ponens, applied one after another, that always take true statements to true statements

A Proof System for Propositional Logic



$$\frac{}{\Gamma, \Phi \longrightarrow \Phi}\,(Ax)$$

$$\frac{\Gamma \longrightarrow \Phi \qquad \Gamma \longrightarrow \Phi'}{\Gamma \longrightarrow \Phi \wedge \Phi'}\,(\wedge I)$$

$$\frac{\Gamma \longrightarrow \Phi_1 \wedge \Phi_2}{\Gamma \longrightarrow \Phi_i}\,(\wedge E_i) \quad (i = 1, 2)$$

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Gentzen's Natural Deduction System

- Search in the space of logical statements
- Γ: set (conjunction) of formulas ("assumptions" or "theory")
- $\frac{A}{B}$: From A we can derive B
- Proof systems make it easy to construct proofs without errors

A Proof System for Propositional Logic

- The Axiom rule. If a statement is already one of our assumptions, we may write it down as something we have derived
- Example. We have assumed "it is raining" as one of our facts. Since it is already an assumption, we get to derive "it is raining" directly

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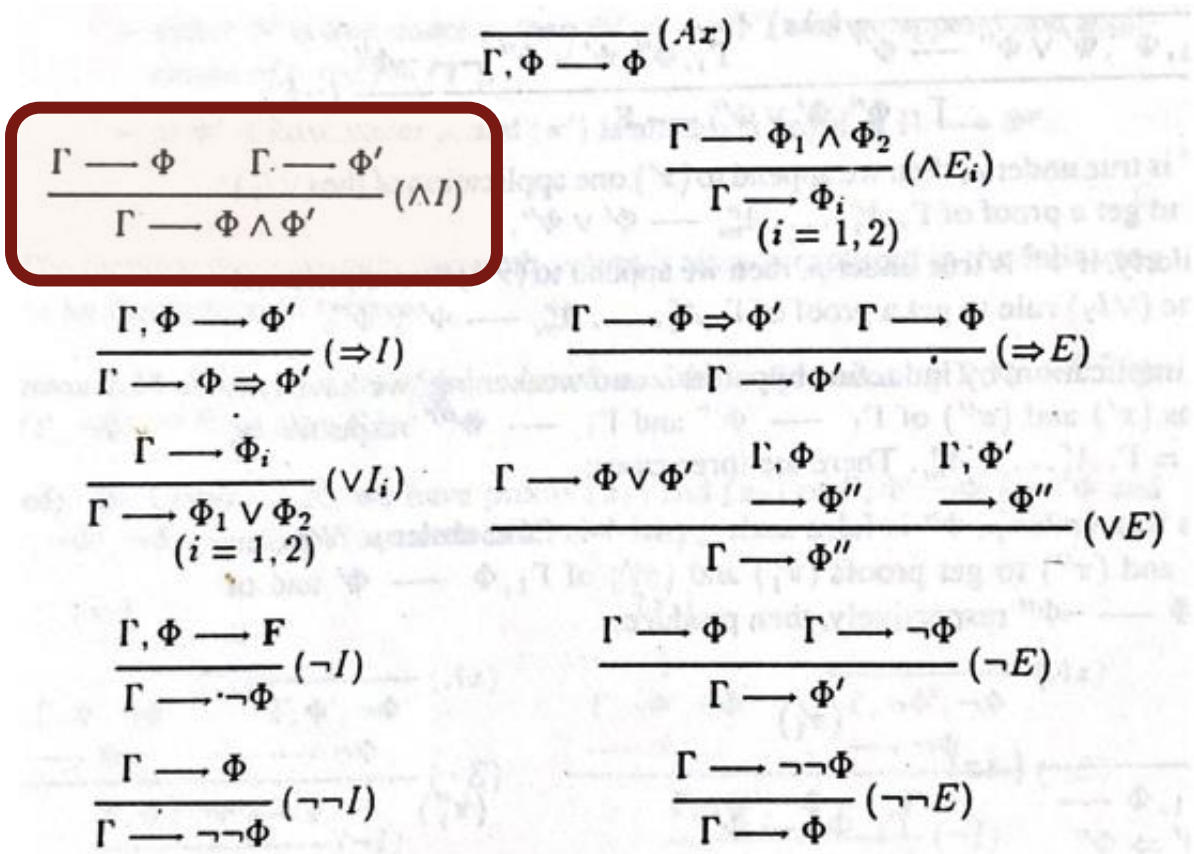
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A Proof System for Propositional Logic

- AND Introduction. If we have P and we have Q, we get to add P and Q

- Example. We have "it is raining" and "it is a weekday", so we get "it is raining and it is a weekday"



A Proof System for Propositional Logic

- AND Elimination. If we have P and Q, we get to add P alone, or Q alone
- Example. We have "Chennai is hot and IIT Madras is strong in AI", so we get "IIT Madras is strong in AI" on its own

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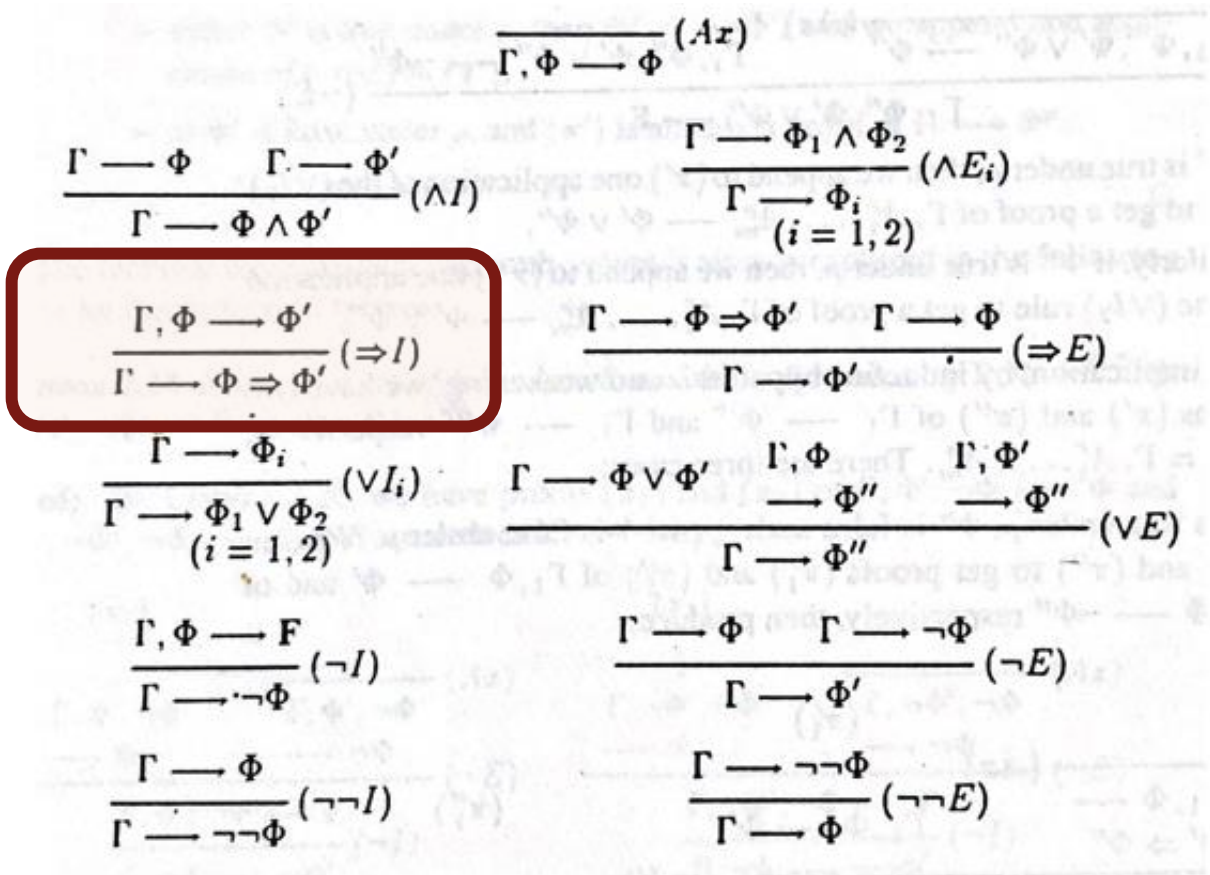
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A Proof System for Propositional Logic

- Implication Introduction. If, after temporarily assuming P, we manage to derive Q, we get to add "P implies Q", and we are allowed to drop the assumption
- Example. Assume "it rains". Using other facts we derive "the ground is wet". We conclude "if it rains, the ground is wet", without assuming rain anymore



A Proof System for Propositional Logic

- Implication Elimination, also called Modus Ponens. If we have P, and we have P implies Q, we get to add Q
- Example. We have "it rains" and "if it rains the ground is wet", so we get "the ground is wet"

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A Proof System for Propositional Logic

- OR Introduction. If we have P, we get to add "P or Q" for any Q we like
- Example. We have "it is raining", so we get "it is raining or it is a national holiday"

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A Proof System for Propositional Logic

- OR Elimination. If we have "P or Q", and P alone lets us derive some R, and Q alone also lets us derive that same R, we get to add R
- Example. We know "today is Saturday or Sunday". If it is Saturday the office is closed, and if it is Sunday the office is closed too, so either way we get "the office is closed"

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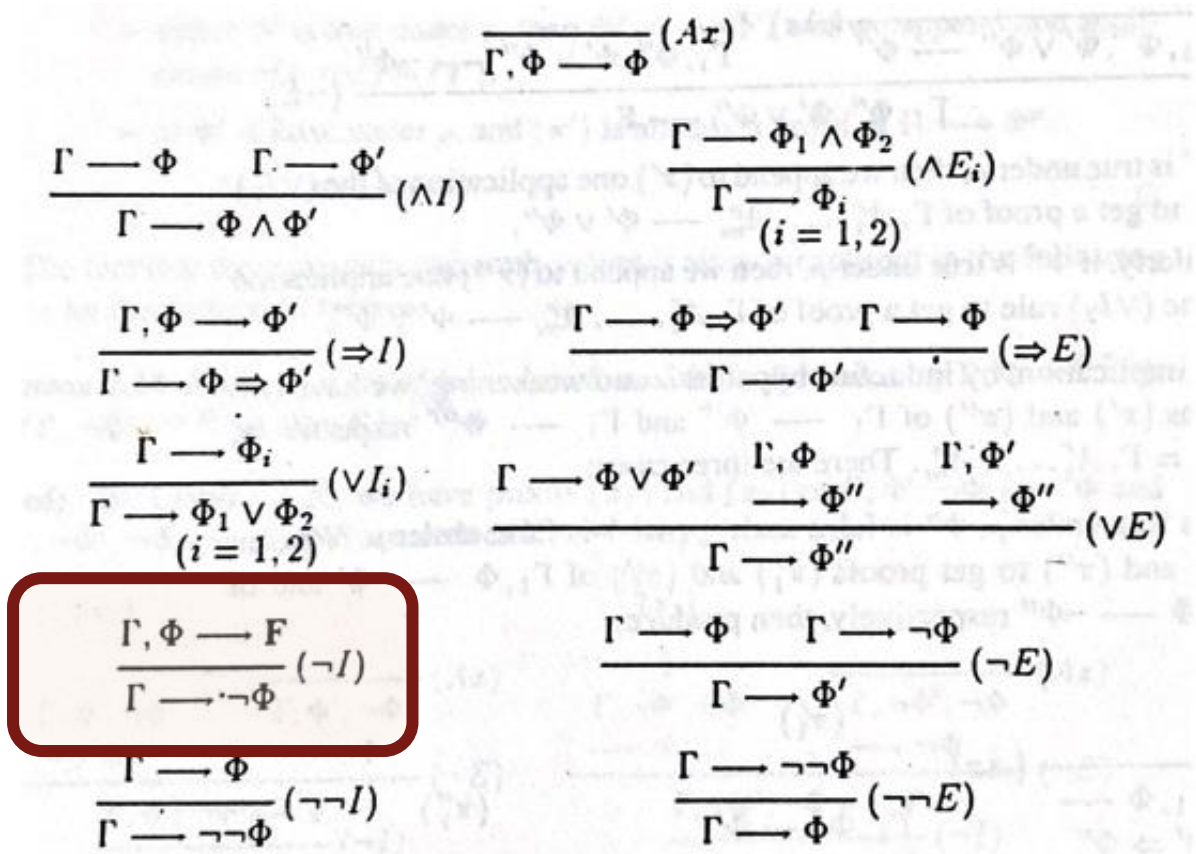
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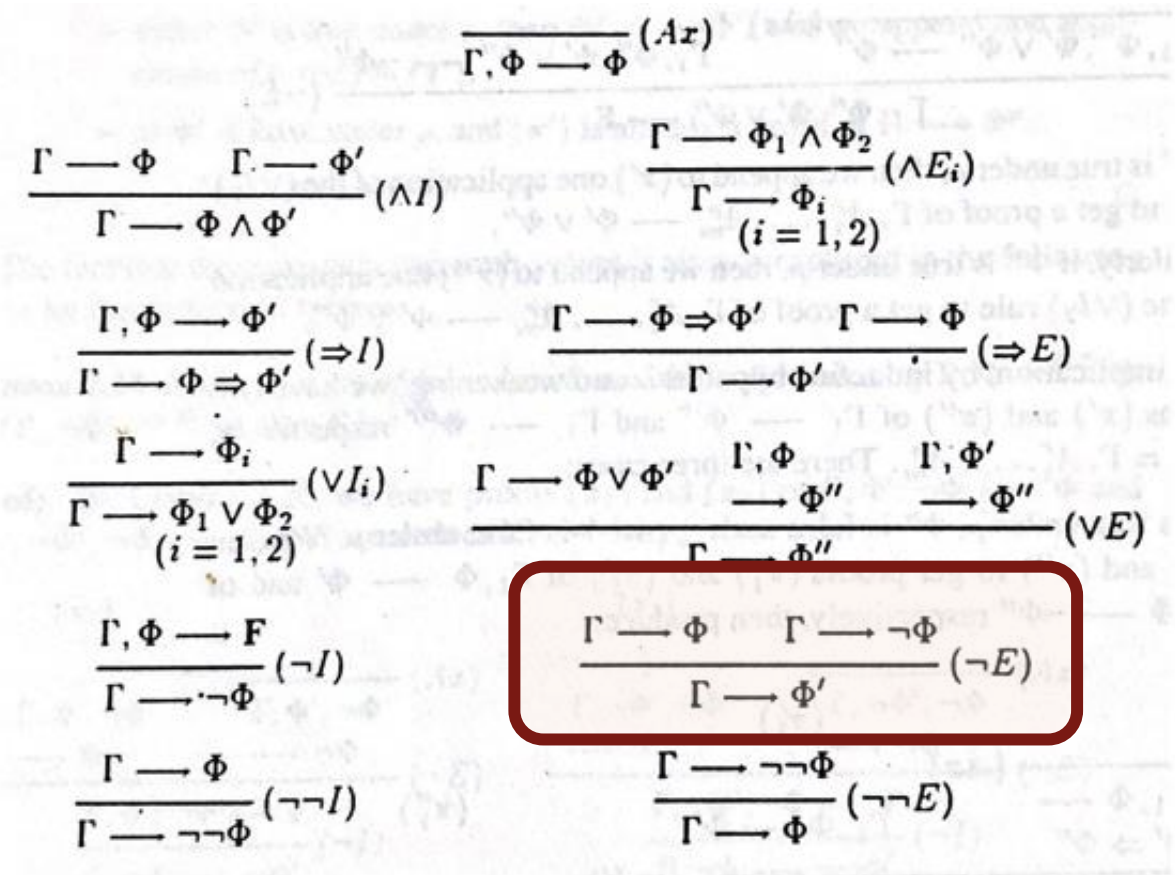
A Proof System for Propositional Logic

- NOT Introduction. If assuming P lets us derive a contradiction, we get to add "not P"
- Example. Assume "the shuttle is on time". This contradicts what we already know, everyone is late today. We conclude "the shuttle is not on time"



A Proof System for Propositional Logic

- NOT Elimination. If we have P and we also have "not P", we get to add anything at all, since a contradiction proves everything
- Example. We have "it is raining" and we also have "it is not raining". From that contradiction, we get to add, for instance, "the moon is made of cheese"



A Proof System for Propositional Logic

- Double Negation Introduction. If we have P, we get to add "not not P"
- Example. We have "it is a weekday", so we get "it is not the case that it is not a weekday"

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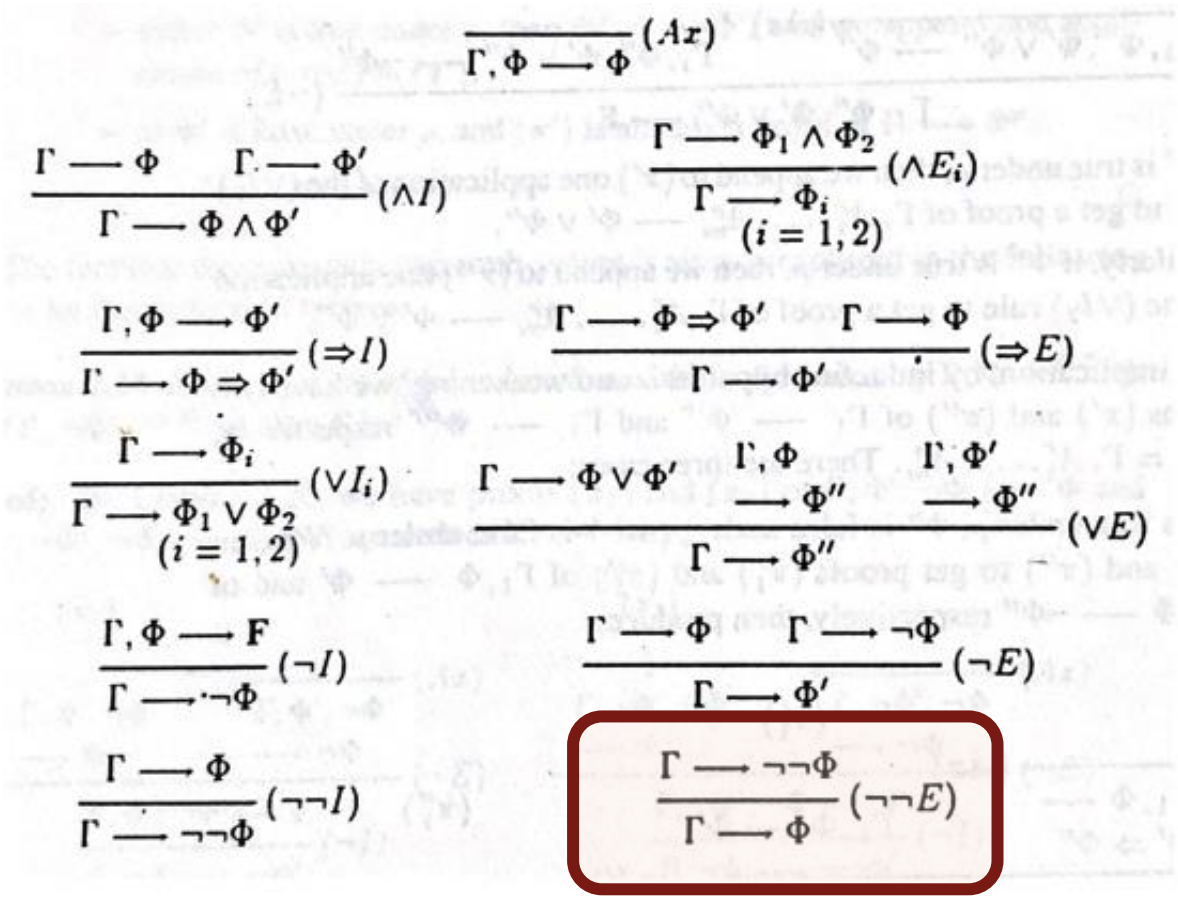
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A Proof System for Propositional Logic

- Double Negation Elimination. If we have “not not P”, we get to add P
- Example. We have “it is not the case that the library is not open”, so we get “the library is open”



Putting the Rules to Work

- Suppose we wish to derive "P implies, not P implies Q" without any assumptions at all
- Start from the desired statement and work upwards. Search among the eleven rules for one that applies
- Using the Axiom rule, NOT Elimination, and Implication Introduction applied twice, we build a complete derivation from nothing up to the full goal statement

